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Non-relativistic anyons, exotic galilean symmetry and noncommutative plane

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Abstract: We show that the Lukierski et al. model, invariant with respect to the two-fold centrally extended Galilei group, can be decomposed into an infinite number of independent copies (differing in their spin) of the "exotic" particle of Duval et al. The difference between the two models is found to be sensitive to electromagnetic coupling. The nature of the noncommutative plane coordinates is discussed in the light of the exotic galilean symmetry. We prove that the first model, interpreted as describing a non-relativistic anyon, is the non-relativistic limit of a particle with torsion related to relativistic anyons.

Keywords: Space-Time Symmetries, Anyons, Non-Commutative Geometry.

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As it has been known for some time, the planar Galilei group admits an "exotic" two-parametric central extension. Recently, two classical systems have been presented that exhibit this extended galilean symmetry. One of them, put forward by Lukierski, Stichel and Zakrzewski [1], uses the second-order lagrangian

$$L_{LSZ} = \frac{1}{2}m\dot{\vec{x}}^2 + \frac{\kappa}{2}\varepsilon_{ij}\dot{x}_i\ddot{x}_j\,, \tag{1}$$

where m, the mass, and κ, the "exotic" parameter, label the central extension [2]. This system requires a 6-dimensional phase space.

The other model [3, 4, 5] is derived from the "exotic" Galilei group following Souriau [6], who identifies classical "elementary" systems with the coadjoint orbits of the group, endowed with their canonical symplectic structures. These orbits are 4-dimensional, and depend on 4 parameters denoted by s, h_0, m and κ. Their symplectic structure induces on the 5-dimensional "evolution space" made of positions, momenta, and time [6] the closed 2-form

$$\omega = d\vec{p}\wedge d\vec{x} + \frac{\kappa}{2m^2}\varepsilon_{ij}dp_i\wedge dp_j - \frac{\vec{p}\cdot d\vec{p}}{m}\wedge dt\,. \tag{2}$$

This (free) model is equivalent to that of a particle in the non-commutative plane [7, 8, 9] when κ/m^2 is identified (as always in what follows) with the "non-commutative parameter" θ [5]. Coupled to a gauge field, the "exotic" model (2) has been used to derive the ground states of the Fractional Quantum Hall Effect [5].

The two-form (2) depends only on m and κ; the two other parameters show up only in the way the Galilei group acts on the orbit, i.e., in the associated conserved quantities. These latter are found to be the momentum, $\vec{p}$,

$$\begin{aligned} \mathcal{J} &= \vec{x}\times\vec{p} + \frac{\theta}{2}\vec{p}^{\,2} + s\,, && \text{angular momentum,} \\ \mathcal{K}_i &= mx_i - p_i t + m\theta\varepsilon_{ij}p_j\,, && \text{boosts,} \\ H &= \frac{\vec{p}^{\,2}}{2m} + h_0\,, && \text{energy,} \end{aligned} \tag{3}$$

supplemented with m and $\kappa = m^2\theta$. The parameter s is hence interpreted as *anyonic spin*, and h_0 is the *internal energy*. In [5] $s = h_0 = 0$.

The two models look rather different. Below we show, however, that the model (1) is in fact composed of an infinite number of independent copies (differing in their spin) of the model (2), (3), and point out that the difference between the models is sensitive to electromagnetic coupling. We discuss the nature of the noncommutative plane coordinates in the light of the exotic galilean symmetry, and argue that the model (1) represents *nonrelativistic anyons*. We demonstrate also that the model (1) is in fact the non-relativistic limit of the particle with torsion [10], which underlies the relativistic anyons.

Let us start with introducing Lagrange multipliers p_i and new variables y_i; then, adding the term $p_i(\dot{x}_i - y_i)$ to (1) yields the first-order lagrangian

$$L_I = \vec{p}\cdot\dot{\vec{x}} + \frac{\kappa}{2}\vec{y}\times\dot{\vec{y}} + \frac{m\vec{y}^{\,2}}{2} - \vec{p}\cdot\vec{y} \tag{4}$$

equivalent to (1). Introducing the coordinates

$$X_i = x_i + m\theta\varepsilon_{ij}y_j - \theta\varepsilon_{ij}p_j\,, \qquad Q_i = \theta(my_i - p_i) \tag{5}$$

and p_i on the 6-dimensional phase space allows us to present the symplectic structure and the hamiltonian associated with (1) through (4) as

$$\begin{aligned} \Omega &= dp_i \wedge dX_i + \frac{\theta}{2}\varepsilon_{ij}dp_i \wedge dp_j + \frac{1}{2\theta}\varepsilon_{ij}dQ_i \wedge dQ_j\,, \\ H &= \frac{\vec{p}^2}{2m} - \frac{1}{2m\theta^2}\vec{Q}^2. \end{aligned} \tag{6}$$

Our coordinates $\vec{X}$ are similar to but still different from those, $\vec{X}^L$, of Lukierski et al.; in fact $X_i = X_i^L - \theta\varepsilon_{ij}p_j$. Note that the "internal" coordinates, $\vec{Q}$, measure the amount by which the "external" momentum, $\vec{p}$, differs from (m-times) the velocity, $\vec{y} = \dot{\vec{x}}$. In the coordinates (5), the "external" and "internal" motions, described by $\vec{X}$ and $\vec{Q}$, respectively, are decoupled:

$$\dot{X}_i = \frac{p_i}{m}\,, \qquad \dot{p}_i = 0\,, \qquad \dot{Q}_i = \frac{1}{m\theta}\varepsilon_{ij}Q_j\,. \tag{7}$$

Thus, while the external motion is free, $\ddot{\vec{X}} = 0$, the internal coordinates perform a uniform rotation with angular velocity $(m\theta)^{-1}$. By (5), the original coordinates $x_i = X_i - \varepsilon_{ij}Q_j$ rotate hence, in general, around the uniformly translating guiding center, c.f. [1].[1] Such a "Zitterbewegung" would be most surprising for a free non-relativistic particle, and we argue that it is the new coordinate $\vec{X}$, and not the original ones, $\vec{x}$, that should be viewed as physical. Note, however, that while original coordinates commute, $\{x_i, x_j\} = 0$, both the "external" and the "internal" positions, $\vec{X}$ and $\vec{Q}$, respectively, are non-commuting,

$$\{X_i, X_j\} = \theta\varepsilon_{ij}, \qquad \{Q_i, Q_j\} = -\theta\,\varepsilon_{ij}. \tag{8}$$

The action of the Galilei group on $\vec{X}$ and $\vec{p}$, deduced from the natural action using (5), is conventional. The internal coordinate $\vec{Q}$ is left invariant in turn by the Galilei boosts, the rotations act on it as on a vector, $\vec{Q} \to R\vec{Q}$. The system is invariant with respect to this action, allowing us to recover the conserved quantities (equivalent to those in [1]), namely $\vec{p}$, augmented with

$$\begin{aligned} \mathcal{J} &= \vec{X} \times \vec{p} + \frac{\theta}{2}\vec{p}^2 + \frac{1}{2\theta}\vec{Q}^2\,, \\ \mathcal{K}_i &= mX_i - p_i t + m\theta\varepsilon_{ij}p_j\,, \end{aligned} \tag{9}$$

and the energy, H, in (6). This latter can also be represented as

$$H = \frac{2}{m}\vec{p}^2 - \frac{1}{m\theta}(\mathcal{J} - m^{-1}\varepsilon_{ij}\mathcal{K}_i p_j)\,. \tag{10}$$

This means that like in the case of the usual free non-relativistic spinless particle ($\kappa = 0$), the energy is defined in terms of other generators of the Galilei group. Let us also observe that the coordinates X_i form a *galilean vector* since they have the correct transformation property under boosts, namely $\{\mathcal{K}_i, X_j\} = -\delta_{ij}t$.

[1] It is interesting to observe the analogy with the motion of a usual spinless charged particle in combined constant magnetic and electric fields, though the particle here is free.

For comparison with [1], we mention that though in the coordinates X_i^L the boosts appear in a more simple form, $\mathcal{K}_i = mX_i^L - p_i t$, their brackets are different from (8) in sign, $\{X_i^L, X_j^L\} = -\theta\varepsilon_{ij}$, and unlike our X_i, they satisfy the relation $\{\mathcal{K}_i, X_j^L\} = -\delta_{ij}t - m\theta\varepsilon_{ij}$, which means that they *do not form* a galilean vector.

Our clue is to observe that the system (6) can consistently be restricted to the surface

$$S_C: \quad \vec{Q}^2 - C^2 = 0\,, \qquad C = \text{const}\,. \tag{11}$$

For $C = 0$, in particular, S_0 is a 4-*dimensional* surface: the internal motion is reduced to a single point. Then the restriction of (6) yields the symplectic form $\Omega_0 = dp_i \wedge dX_i + (\theta/2)\varepsilon_{ij}dp_i \wedge dp_j$ and the hamiltonian $H_0 = \vec{p}^2/2m$ respectively, so that $\Omega_0 - dH_0 \wedge dt$ is the "exotic" 2-form (2). The coordinates $\vec{X}$ are reduced now in fact to the original ones that perform the usual free motion with *no whirling* around, and can, therefore, be viewed as physical. The conserved quantities (9) become, furthermore, precisely those found in [5], i.e. (3) with no anyonic spin and vanishing internal energy, $s = 0$ and $h_0 = 0$, respectively. Note also that, consistently with (8), the original coordinates x_i become also non-commuting, owing to the constraint (11). This result is understood by observing that the action of the Galilei group on 6-dimensional phase space leaves the surfaces S_C invariant. For $C = 0$ it is transitive as well as symplectic. By Souriau's theorem [6] S_0, endowed with the restricted two-form Ω_0, is hence equivariantly symplectomorphic to a coadjoint orbit of the group, namely to ω in (2) and (3), with $s = h_0 = 0$. Note also that when $\vec{Q}^2 = 0$, the momentum is m times the velocity, $\vec{p} = m\dot{\vec{x}}$. The second-order term in (1) becomes the "exotic" term $(\theta/2)\varepsilon_{ij}p_i\dot{p}_j$, and the lagrangian (4) reduces to the one used in [5].

Let us now turn to the 5-dimensional surfaces S_C with $C \neq 0$. The coordinates $\vec{X}$ move still freely but the original coordinates, $\vec{x}$, suffer the Zitterbewegung found above. An interesting insight can be gained by studying in the variational aspects of the system. The two-form and hamiltonian in (6) correspond indeed to the variational (Cartan [6]) 1-form (consistent with (4)),

$$\Gamma = \vec{p}\cdot d\vec{X} + \frac{\theta}{2}\vec{p}\times d\vec{p} + \frac{1}{2\theta}\vec{Q}\times d\vec{Q} - \left(\frac{\vec{p}^2}{2m} - \frac{1}{2m\theta^2}\vec{Q}^2\right)dt\,. \tag{12}$$

Its restriction to S_C is plainly

$$\gamma = \vec{p}\cdot d\vec{X} + \frac{\theta}{2}\vec{p}\times d\vec{p} + \frac{C^2}{2\theta}d\sigma - \left(\frac{\vec{p}^2}{2m} - \frac{1}{2m\theta^2}C^2\right)dt\,, \tag{13}$$

where we have parametrized $\mathcal{Q} = Q_1 + iQ_2$ as $\mathcal{Q} = Ce^{i\sigma}$. Note that there is no equation of motion for the residual internal variable σ which became now unphysical.

The action of the Galilei group is consistent with the constraint, so it is still a symmetry, yielding once again the conserved quantities (9) with $\vec{Q}^2$ replaced by the constant C^2. We obtain hence the shifted moment map [6] (3) with anyonic spin $s = C^2/2\theta$ and internal energy $h_0 = -C^2/2\theta^2$, whose presence comes from that, when our $\vec{X}$ is identified with the physical coordinate in (2), then (13) only differs from the variational 1-form used in [5] in the (exact) "internal" 1-form $(C^2/2\theta)d\sigma$ and in a shift in the hamiltonian. Note that, curiously, the internal degrees of freedom, $\vec{Q}$, contribute a *negative* term to the energy.

Physically, the restriction to S_C yields non-relativistic *anyons.*

The exterior derivative of the variational 1-form (13) to the odd dimensional manifold S_C is singular; its kernel corresponds to σ. Factoring this out, the residual $4D$ space can be parametrized by $\vec{X}$ and $\vec{p}$, upon which the (extended) Galilei group acts transitively, so that Souriau's theorem [6] identifies it, once again, with a coadjoint orbit with symplectic structure (2).

Note that the spin and the internal energy derived above are linked, $h_0 = -(m\theta)^{-1}s$. It is rather easy to modify the original model [1] to make them independent: it is enough to add to L_{LSZ} the spin term

$$L_s = s_0\varepsilon_{ij}\frac{\dot{x}_i\ddot{x}_j}{\dot{\vec{x}}^2}\,. \tag{14}$$

This term is in fact $s_0 d\varphi$, where φ is the polar angle of the planar velocity vector $\dot{\vec{x}}$. This "Aharonov-Bohm-type" term is topological: it merely changes the action by $2\pi s_0\times$(winding number), where the winding number labels the homotopy class i.e. counts the number of times the velocity space curve winds around the origin. It does not change the equations of motion (although it contributes at the quantum level). Taking into account the way the Galilei group acts, is simply results in adding s_0 to the conserved angular momentum, yielding the general case in (3). Note also that the exact "internal" term $(C^2/2\theta)d\sigma$ in (13) is actually also of a similar form: it is indeed $(C^2/2\theta)\varepsilon_{ij}Q_i\dot{Q}_j/\vec{Q}^2$.

Let us mention that from the 1-form (12) and the relation (11), the technique of Dirac constraints allows us to derive the lagrangian

$$L_C = m\vec{y}\cdot\dot{\vec{x}} - \frac{m\vec{y}^2}{2} + \frac{\kappa}{2}\vec{y}\times\dot{\vec{y}} - mC|\dot{\vec{x}}-\vec{y}|\,, \tag{15}$$

for which (11) appears as a first class constraint. The latter generates gauge transformations $\sigma\to\sigma+\alpha(t)$ which mean that the circular coordinate σ is a pure gauge variable. As a result, lagrangian (15) corresponds to the reduced system. Note that for any value of C, the coordinates X_i, unlike x_i, are the gauge-invariant variables (commuting with the first class constraint (11)), and that for $C=0$, (15) reduces to the lagrangian of the model (2) under the identification $\vec{p}=m\vec{y}$, c.f. (13).

By virtue of the decomposition into external and internal spaces, the quantization of the model is straightforward. The external space is conveniently quantized using the *commuting* position-like variables

$$\tilde{X}_i = X_i + \frac{\theta}{2}\varepsilon_{ij}p_j\,, \tag{16}$$

yielding the quantized operators at once [5]. Let us stress, however, that like the coordinates X_i^L, the commuting variables $\tilde{X}_i$ are *not* physical, since they transform incorrectly under galilean boosts, namely as $\{\mathcal{K}_i,\tilde{X}_j\} = -\delta_{ij}t - (m\theta/2)\varepsilon_{ij}$. So, the coordinates X_i describing the noncommutative plane are identified uniquely as the coordinates forming a galilean vector.

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The internal space is the symplectic plane; its quantization is hence conveniently achieved using the " Bargmann-Fock" framework, which yields the internal wave functions $f(\mathcal{Q})e^{-|\mathcal{Q}|^2/4}$ where $f(\mathcal{Q})$ is holomorphic. The internal hamiltonian, $H_{int} = -\vec{Q}^2/2m\theta$, can be viewed as the hamiltonian of a 1-dimensional oscillator with *phase space* coordinate $\mathcal{Q}$, *negative* mass $-m$ and frequency $|m\theta|^{-1}$. Its spectrum is, therefore, $E_{int} = -(m|\theta|)^{-1}(n+1/2)$, c.f. [1]. The divergent negative energies can be eliminated by constraining the system to S_C, which only leaves us with a constant (negative) shift; requiring positivity [1] amounts to setting $C=0$.

Since only the rotations act effectively on the internal space, only this latter contribute, namely adding $s=-m\theta E_{int}$ to the conserved angular momentum. When restricted to S_C, the radial oscillations disappear and s becomes the (constant) anyonic spin. Let us stress that $s=C^2/2\theta$ can take any real value.

When we put the system into an electromagnetic field, the result crucially depends on the way the coupling is defined [5]. Lukierski et al. only consider the coupling to a scalar potential. Their rather strange-looking, velocity-dependent expression is chosen so that, when expressed in their " good" coordinates $\vec{X}^L$, it becomes simply $V(\vec{X}^L)$. Here we posit instead our coupling so that it becomes natural when expressed in our coordinates $\vec{X}$. Then, in the case of the scalar potential coupling, the quantization rule (16) results in the Moyal product term in the Schrödinger equation,

$$V(X)\Psi(\tilde{X}) = V\left(\tilde{X}_i - \frac{\theta}{2}\varepsilon_{ij}p_j\right)\Psi(\tilde{X}) = V(\tilde{X}) \star \Psi(\tilde{X})\,, \tag{17}$$

that reproduces correctly the structure of the one-particle sector of the quantum theory on the noncommutative plane [9, 8, 11].

Now, there is no difficulty to include a magnetic field: we simply apply Souriau's prescription [6] who suggests to add to the exterior derivative of the free variational form (12) the electromagnetic two-form $F=(1/2)F_{\mu\nu}(X)dX^\mu\wedge dX^\nu$, $X^\mu=(t,\vec{X})$, expressed in the "natural" position coordinates. This coupling is plainly gauge invariant and, (unlike for the rule proposed in [7]), the Jacobi identities hold for non-constant fields, provided F merely satisfies the homogeneous Maxwell equations, $dF=0$. Applying this rule yields unchanged internal motion; as to the external motion, we recover the model studied in detail in [5], which, in the critical case $eB\theta=1$, yields the ground states of the Fractional Quantum Hall Effect. Note also that the coupling of relativistic anyons to gauge fields have been considered in [17].

Here, we confine ourselves with presenting how this coupling rule — natural in our framework — appears in the context of the model [1]. In terms of the coordinates x_i, y_i and p_i used in (4) we get, for a constant magnetic field B (in the symmetric gauge), the rather complicated-looking phase-space lagrangian

$$\begin{aligned} L_B = p_i\left(\frac{m^*}{m}\dot{x}_i - y_i\right) + \frac{m\vec{y}^2}{2} + \frac{m^2\theta}{2}\varepsilon_{ij}y_i\dot{y}_j - \\ - \frac{e\theta^2 B}{m}\varepsilon_{ij}p_i\dot{y}_j + \frac{eB}{2}\varepsilon_{ij}(x_i + m\theta\varepsilon_{ik}y_k)(\dot{x}_j + m\theta\varepsilon_{il}\dot{y}_l) + \frac{e\theta^2}{2}B\varepsilon_{ij}p_i\dot{p}_j\,, \end{aligned} \tag{18}$$

where $m^* = m(1 - eB\theta)$ is the effective mass term introduced in [5]. This formula, hopeless as it is, reveals nonetheless at least two essential points. Firstly, putting the effective mass to vanish, $m^* = 0$, switches off the kinetic term; the system becomes singular, necessitating the reduction procedure described in [5]. Secondly, the appearance of $\dot{p}_i$ in the last term implies that the momentum *ceases to be a Lagrange multiplier* and becomes rather fully dynamical. Therefore, it can not come by Legendre transformation from any lagrangian expressed in terms of $\vec{x}$ and its derivatives alone.

Other consistent couplings are also possible, though. One could add naively, for example, the standard constant magnetic term to the original lagrangian, i.e., replace L_{LSZ} by

$$\tilde{L}_B = \frac{1}{2} m \dot{\vec{x}}^2 + \frac{\kappa}{2} \varepsilon_{ij} \dot{x}_i \ddot{x}_j + \frac{eB}{2} \varepsilon_{ij} x_i \dot{x}_j \,. \tag{19}$$

This would yield three types of motions, depending on the sign of

$$\mu^2 = 1 - 4eB\theta \,. \tag{20}$$

For $\mu^2 > 0$ we get in fact a superposition of uniform circular motions with frequencies $\omega_\pm = -(2m\theta)^{-1}(1 \pm \mu)$. For $\mu^2 < 0$, $z = x_1 + ix_2$ moves according to some superposition of $\exp[-i(2m\theta)^{-1}t] \exp[\pm|\mu(2m\theta)^{-1}|t] z_0^\pm$, which are rotational motions with increasing or decreasing radii. For $\mu^2 = 0$ we get finally a superposition of a uniform, constant-radius rotation, $\exp[-i(2m\theta)^{-1}t] z_0$, with a rotation $t \exp[-i(2m\theta)^{-1}t] \tilde{z}_0$ whose radius is linear in time. None of the motions with $\mu^2 \leq 0$ appears to be physical; such a coupling has therefore a limited interest. Note that the critical value $\mu = 0$ has appeared before in a related but slightly different context [16].

Since the higher-derivative model (1) has been interpreted as describing non-relativistic anyons, we arrive at the natural question of its relation to the higher derivative model of the particle with torsion [10], which underlies relativistic anyons and whose action reads

$$\mathcal{A} = \int (-m + \alpha \varrho) ds \,, \qquad \varrho = \frac{\epsilon^{\mu\nu\lambda} x'_\mu x''_\nu x'''_\lambda}{x''^2} \,, \tag{21}$$

where the prime means derivation with respect to arc length, $x'_\mu = dx_\mu/ds$, $ds^2 = -dx^\mu dx^\nu \eta_{\mu\nu}$, $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1)$, and $\epsilon^{0ij} = \varepsilon_{ij}$.

The action (21) with $\alpha = 1/2$ appeared originally in a euclidean version in the context of the Bose-Fermi transmutation mechanism [12]. Like the model (1), the system (21) possesses dynamical spin degrees of freedom, and reveals the Zitterbewegung. In analogy with the possible energy values $E > 0$, $E = 0$ and $E < 0$ of the hamiltonian (6), the model (21) possesses massive, massless, and tachyonic solutions, and its classical motions are similar to those of the $(2+1)$D relativistic charged scalar particle moving in constant electric and magnetic fields [13]. Finally, it was found that the reduction of the model (21) to the surface of constant spin results in a relativistic anyon model [10, 14, 15]. Such a formal analogy is, of course, not accidental, and below we prove that the model (1) is the non-relativistic limit of the relativistic model (21).

In an arbitrary parametrization $x_\mu = x_\mu(\tau)$, the structure of the equations of motion for the system (21),

$$\dot{p}_\mu = 0 \,, \qquad p_\mu = \frac{m}{\alpha} J_\mu - \frac{1}{\alpha q} \epsilon_{\mu\nu\lambda} J^\nu \dot{J}^\lambda \,, \qquad J_\mu = \frac{\alpha}{q} \dot{x}_\mu \,, \qquad q = \sqrt{-\dot{x}^2} \,, \tag{22}$$

is similar to that of the equations (7) written in the initial coordinates $\vec{x}$,

$$\dot{\vec{p}} = 0\,, \qquad p_i = my_i + \theta m^2 \varepsilon_{ij}\dot{y}_j\,, \qquad \vec{y} = \dot{\vec{x}}\,. \tag{23}$$

Note that on the surface defined by the equations of motion (22), the torsion of *the world trajectory* is reduced to the constant, $\varrho = -m\alpha^{-1}$, whereas its curvature, k, $k^2 = x''_\mu x''^\mu$, is conserved, $k^2 = \alpha^2(m^2 + p^2)$. Identifying in (22) the evolution parameter with x^0 (laboratory time gauge), $x^0 = \tau$, and taking a non-relativistic limit, $|\dot{\vec{x}}| \ll 1$, we immediately find that the equations (22) with $\mu = i$ are reduced exactly to (23) when $-\alpha$ is identified with the "exotic" parameter κ.

To find the non-relativistic lagrangian of the model (21), let us turn to the canonical formalism. In accordance with (22), the system (21) is described by the constraints

$$\chi \equiv p_\mu J^\mu + \alpha m = 0, \qquad \psi \equiv P_q = 0\,, \tag{24}$$

responsible for reparametrization invariance. The symplectic form and the hamiltonian are

$$\begin{aligned} \Omega_{tor} &= dP_q \wedge dq + dp_\mu \wedge dx^\mu + \frac{1}{2}\alpha^{-2}\epsilon_{\mu\nu\lambda}J^\mu dJ^\nu \wedge dJ^\lambda\,, \\ H_{tor} &= \alpha^{-1}q\chi + w\psi\,, \end{aligned} \tag{25}$$

where $w = w(\tau)$ is an arbitrary function [10]. The J_μ take values in the two-sheeted hyperboloid, $J^2 = -\alpha^2$, and generate the $sl(2,R)$ algebra, $\{J_\mu, J_\nu\} = -\epsilon_{\mu\nu\lambda}J^\lambda$. The angular momentum vector

$$\tilde{\mathcal{J}}_\mu = -\epsilon_{\mu\nu\lambda}x^\nu p^\lambda + J_\mu \tag{26}$$

is an integral of the motion of the system (21) and forms, together with the energy-momentum vector p_μ, the Poincaré algebra $iso(2,1)$. The first constraint from (24), being the classical analog of the quantum (2+1)-dimensional Majorana equation, has three types of solutions mentioned above. In the massive ($p^2 < 0$) and the tachyonic ($p^2 > 0$) sectors it defines the mass-spin relation, where the spin (central Casimir element) of $iso(2,1)$ is $\mathcal{S} = p\tilde{\mathcal{J}}/\sqrt{|p^2|}$. In the hamiltonian picture, the gauge freedom is fixed by imposing two gauge conditions $x^0 - \tau = 0$ and $q - \alpha/J^0 = 0$. Let us focus our attention to the massive sector $p^2 < 0$, and, again, consider the non-relativistic limit $|\dot{\vec{x}}| = |\vec{J}/J^0| \ll 1$. Then the symplectic form Ω_{tor} reduces to $\omega_t = dp_i \wedge dx_i + \frac{1}{2}\alpha^{-1}\epsilon_{ij}dJ_i \wedge dJ_j$. Using the Majorana-type constraint (24), the hamiltonian becomes, in turn, $H_t = p^0 = \alpha^{-1}\vec{p}\cdot\vec{J} - \frac{1}{2}m\alpha^{-2}\vec{J}^2 + m$. Here we have taken into account the explicit dependence of the gauge condition $x^0 - \tau = 0$ on the evolution parameter, and used that the positive energy sector $p^0 > 0$ corresponds to the upper sheet of the hyperboloid with $J^0 = \sqrt{\vec{J}^2 + \alpha^2}$. Identifying $\alpha^{-1}\vec{J}$ with $\vec{y}$, and as before, the $-\alpha$ with the "exotic" parameter κ, and using the relations (5), we find that ω_t is reduced exactly to the symplectic form (6), whereas the hamiltonian takes the form of the hamiltonian in (6) shifted by a constant, $H_t = \vec{p}^2/2m - (2m\theta^2)^{-1}\vec{Q}^2 + m$. This means that the non-relativistic limit of the model (21) is described effectively by the lagrangian $L_t = L_I - m$, where L_I is the first order lagrangian (4).

In the non-relativistic limit, the $iso(2,1)$ generators (26) are reduced to $\tilde{\mathcal{J}}_0 = \mathcal{J} + \theta m^2$, $\tilde{\mathcal{J}}_j = \varepsilon_{ij}\mathcal{K}_j$, where $\mathcal{J}$ and $\mathcal{K}_i$ are integrals of motion (9), whereas the $iso(2,1)$ spin $\mathcal{S}$ is reduced to $\vec{Q}^2/(2\theta) + \theta m^2$. Finally, we note that the non-relativistic limit of squared curvature, $k^2 = (x'')^2$, is $\vec{Q}^2/(m\theta)^4$.

In conclusion, we found that the model of Lukierski et al. [1] can be decomposed into a union of slightly deformed copies of the model studied in [4, 3, 5], which correspond to non-relativistic anyons. Our coordinates $\vec{X}$ transform as vectors under Galilei boosts, just like their non-commuting relativistic counterparts for the anyon considered in [18]. They are analogues of the Foldy-Wouthuysen coordinates for the Dirac particle. The commuting coordinates $\tilde{\vec{X}}$ are in turn analogous to those localizable relativistic coordinates. The latter are, however, not a Lorentz vector [18], and correspond rather to the Newton-Wigner coordinates [19].

Let us stress that the coordinates X_i describing the noncommutative plane have been identified by us uniquely as the coordinates which form a space vector with respect to the transformations associated with the exotic Galilei group generators p_i, $\mathcal{J}$ and $\mathcal{K}_i$ from (9). They are translated by the hamiltonian (10) in contrast with the initial coordinates x_i subjected to a Zitterbewegung.

The additional internal degree of freedom, $\vec{Q}$, can be related as spin, since the conserved angular momentum, $\mathcal{J}$ in (9), reduces, in the rest frame, to $\vec{Q}^2/2\theta$. Remember that $\vec{Q}$ is unaffected by galilean boosts and space translations. In our view the proper motion of $\vec{Q}$ is a gauge artifact; the only physical quantity is $\vec{Q}^2/2\theta$, which is a constant of the motion, and we identify it with the anyonic spin.

Eliminating the negative-energy quantum excitations amounts, furthermore, to suppressing (reducing to a fixed value) this internal spin degree of freedom and yields the minimal system studied in [4, 3, 5]. The word "minimal" refers here to the action of the (extended) Galilei group, which is transitive in [5], but not for the case of the model [1]. In a quantum language, transitivity corresponds to having an irreducible representation. The elimination of the negative energy states, is, however, *not* mandatory for a non-relativistic system. (Just think of the Kepler problem, where the bound motions have negative energy).

We have observed that the interaction described in terms of the covariant coordinates X_i results, at the quantum level, in the Moyal product term (17) in the corresponding Schrödinger equation. We have also pointed out that while the exotic model (2), (3) can be easily coupled to an electromagnetic field, the higher-derivative model (1) does not seem appropriate to accommodate a magnetic field in a natural way.

We have demonstrated that the Lukierski et al. system is the non-relativistic limit of the model of the relativistic particle with torsion [10]. In this limit the quantity $\vec{Q}^2/2\theta$ is identified as a (shifted by a constant) analog of the $iso(2,1)$ spin of the model (21), and, on the other hand, is a rescaled (by a constant) squared curvature of the world trajectory of the particle (21). As it was shown earlier [10, 14, 15], the reduction of the model (21) to the surface of the constant spin value $\mathcal{S} = -\alpha$ produces relativistic anyons, whose non-relativistic limit is, in turn, the model of Duval et al., c.f. [20]. This is in a complete agreement with the relationship of the models (1), (2)-(3) and (21) elaborated here.

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Having established the relation of the non-relativistic model [1] with its relativistic counterpart [10], we note that the association of the indefinite metric with higher-derivative term, as it was done in one of the alternative quantization schemes in [1], does not seem natural. Indeed, analogously to the relativistic case, the indefinite metric prescription is not consistent with the real nature of the classical spin variables [10]. On the other hand, it is known [21] that the presence of higher derivatives in lagrangian in relativistic case does not obligatorily leads to the appearance of the tachyonic states in the spectrum.

At last, we note that the problem of introducing interaction discussed here in the context of non-relativistic anyons could give a new perspective for analogous problem in the relativistic case [17]. The obtained results could also be helpful for clarifying the question of spin-statistics relation, which still remains open in the group-theoretical approach to anyons [18].

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